

Ordered Factorizations with k Factors

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Abstract

We give an overview of combinatoric properties of the number of ordered k -factorizations $f_k(n, l)$ of an integer, where every factor is greater or equal to l . We show that for a large number k of factors, the value of the cumulative sum $F_k(x, l) = \sum_{n \leq x} f_k(n, l)$ is a polynomial in $\lfloor \log_l x \rfloor$ and give explicit expressions for the degree and the coefficients of this polynomial. An average order of the number of ordered factorizations for a fixed number k of factors greater or equal to 2 is derived from known results of the divisor problem.

1 Introduction

We study the number of ordered factorizations $f_k(n, l) = \#\{(i_1, \dots, i_k) \geq l, i_1 \cdots i_k = n\}$ of a positive integer n with exactly k factors greater or equal to l , where factorizations with the same factors in different orders are considered to be different. Here $\#\{\cdots\}$ denotes the cardinality of a set. For example, for $n = 12$, $l = 2$ and $k = 1, 2, 3$ we have

$$\begin{aligned} f_1(12, 2) &= 1 = \#\{(12)\} \\ f_2(12, 2) &= 4 = \#\{(2, 6), (6, 2), (3, 4), (4, 3)\} \\ f_3(12, 2) &= 3 = \#\{(2, 2, 3), (2, 3, 2), (3, 2, 2)\}. \end{aligned}$$

We are mainly interested in the cases $l = 1$ and $l = 2$, but some properties rely on the recursive structure of the functions $f_k(n, l)$ for $l > 2$ (see theorem 2.1 below), so that it is useful to treat the minimal admissible value l for the factors as a separate parameter. In some studies, cf. [6] or [18] for example, the set of admissible factors is further constrained, but we restrict ourselves to the case of factors greater or equal to a minimal value l .

To simplify the notation, we omit the parameter l for $l = 1$ and $l = 2$ and use the notations $d_k(n) := f_k(n, 1)$ and $f_k(n) := f_k(n, 2)$. We denote the corresponding summatory functions with capital letters and write $F_k(x, l) := \sum_{n \leq x} f_k(n, l)$, $F_k(x) := \sum_{n \leq x} f_k(n)$ and $D_k(x) := \sum_{n \leq x} d_k(n)$ for real $x \geq 1$.

Properties of ordered factorizations have a long history in the mathematical literature. We refer to [10] and [9, section 4] for good overviews.

An explicit formula for $f_k(n)$ was given by MacMahon in [14], compare also [11]. If the prime factorization of an integer n is given by $n = p_1^{e_1} p_2^{e_2} \cdots p_{\omega(n)}^{e_{\omega(n)}}$, where $\omega(n)$ denotes the number of

distinct prime factors of n , MacMahon's explicit formula is given by

$$f_k(n) = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{\omega(n)} \binom{e_j + k - i - 1}{e_j}. \quad (1.1)$$

This formula in combination with (2.12) below can also be used to calculate $d_k(n)$ explicitly.

Most of the studies of ordered factorizations focus on the cumulative function $f(n) := \sum_{k=1}^{\infty} f_k(n)$ counting all ordered factorizations, also called the Kalmar function. Kalmar in [8] proved an asymptotic of the form

$$F(x) := \sum_{n \leq x} f(n) = Kx^{\rho} + \Delta(x), \quad (1.2)$$

where the parameters of the main term are given by $\rho = \zeta^{-1}(2) \approx 1.7286$ and $K = -(\rho\zeta'(\rho))^{-1} \approx 0.31817$ and $\zeta(\cdot)$ denotes the Riemann zeta function. The order of the error term in (1.2) has been improved in several steps, the currently best known result is given in [5].

Lower and upper bounds for $f(n)$ are studied in [1], [2] and [9]. In [3] results are given for f -champions, i.e. integers N for which $f(N) > f(n)$ for all $n < N$.

The functions $f_k(n)$ resp. $F_k(x)$ are explicitly treated in [5], [6] and [12]. In [6] a central limit theorem for $F_k(x, l)$ for $x \rightarrow \infty$ is proven¹. Results on the average order of $f_k(n)$ for $k \geq 2$ are given in [5] and [12]. We come back to these results in section 4 below.

It is worth mentioning that the functions $f_k(n)$ and $F_k(x)$ are directly connected to some of the most important arithmetical functions. We denote by $\mu(n)$ the Moebius function, by $M(x) = \sum_{n \leq x} \mu(n)$ the Mertens function, by $\Lambda(n)$ the van Mangoldt function and by $\Pi(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\log n}$ the Riemann prime counting function. We have for $n, x \geq 1$ (see [4, chapter 17.2])

$$\mu(n) = \sum_{k=0}^{\log_2 n} (-1)^k f_k(n) \quad (1.3)$$

$$M(x) = \sum_{k=0}^{\lfloor \log_2 x \rfloor} (-1)^k F_k(x) \quad (1.4)$$

$$\frac{\Lambda(n)}{\log n} = \sum_{k=1}^{\log_2 n} \frac{1}{k} (-1)^{k+1} f_k(n) \quad (1.5)$$

$$\Pi(x) = \sum_{k=1}^{\lfloor \log_2 x \rfloor} \frac{1}{k} (-1)^{k+1} F_k(x), \quad (1.6)$$

where the conventions of (2.3) and (2.4) below for values at $k = 0$ are used. From equation (1.4) it follows that the Mertens function at x can be regarded as the surplus of the number of factorizations of integers smaller or equal to x with an even number of factors over the number of factorizations with an odd number of factors.

The aim of this paper is threefold. First, we want to give a systematic overview of the

¹In fact, the result proven in [6] is more general, since it covers factorizations with constraints.

recursive structure of the quantities $F_k(x, l)$ and $f_k(n, l)$. We do not claim that any of the given formulas is new, but a complete overview does not seem to exist in the literature. Recursive formulas are covered in section 2.

In section 3 we exploit the recursive structure of $F_k(x, l)$ to derive explicit polynomial type formulas when the number of factors k is near its maximum value $\lfloor \log_l x \rfloor$, for $l \geq 2$. Our results generalize an observation in [5, section 8].

In section 4 we consider the average order of $f_k(n)$ for fixed k . Although the results given here are straightforward implications of well known asymptotics of the divisor problem and the fact that $D_k(x)$ is the binomial transform of $F_k(x)$ (see (2.10) below), it seems that the resulting average orders for $f_k(n)$ haven't yet been discussed in the literature.

Notations: i, j, k, l, n, m always denote positive integers, x, y, u, v, w real numbers and s, z complex numbers. We write σ_s for the real part of s . As usual, $\lfloor x \rfloor$ denotes the floor function (the greatest integer smaller than x), $\lceil x \rceil$ denotes the ceiling function (the smallest integer greater than x) and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x . The Riemann zeta function is denoted by $\zeta(s)$. We also use the notation $\zeta_l(s) = \sum_{n=l}^{\infty} n^{-s}$ ($\sigma_s > 1$) for the truncated Riemann zeta function. Empty sums are considered to be zero.

2 Combinatoric identities for $f_k(n, l)$ and $F_k(x, l)$

We first note that, since $l^k > n$ for $k > \lfloor \log_l n \rfloor$ or $l > \lfloor \sqrt[k]{n} \rfloor$, we have for $k, l \geq 2$

$$f_k(n, l) = F_k(n, l) = 0 \quad \text{for} \quad k > \lfloor \log_l n \rfloor \quad \text{or} \quad l > \lfloor \sqrt[k]{n} \rfloor. \quad (2.1)$$

We also have $f_k(n, l) = 0$ for $k > \Omega(n)$, where $\Omega(n) \leq \lfloor \log_2 n \rfloor$ denotes the total number of prime factors of n .

For $k = 1$ we have

$$f_1(n, l) = \begin{cases} 0 & \text{for } n < l \\ 1 & \text{for } n \geq l \end{cases}, \quad F_1(x, l) = (\lfloor x \rfloor - l + 1)^+, \quad (2.2)$$

where $y^+ := \max(0, y)$. From the definition it is clear, that for $n, x \geq 1$ and $k, l \geq 1$

$$f_k(n, l) = \sum_{\substack{i=l \\ i|n}}^n f_{k-1}(n/i, l), \quad f_0(n, l) = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases} \quad (2.3)$$

$$F_k(x, l) = \sum_{i=l}^n F_{k-1}(x/i, l), \quad F_0(x, l) = 1. \quad (2.4)$$

For concrete calculations, these recursive expressions are of limited use due to their computational extensiveness. Note that (2.3) can be written as $f_k(n, l) = f_{k-1}(n, l) * f_1(n, l)$, where $*$ denotes Dirichlet convolution. If we denote by $\mathcal{F}_{k,l}(s)$ the Dirichlet generating function of $f_k(n, l)$, it follows that $\mathcal{F}_{k,l}(s) = \mathcal{F}_{k-1,l}(s)\zeta_l(s)$ and therefore, for $k, l \geq 1$ and $\sigma_s > 1$ (compare,

for example [6])

$$\mathcal{F}_{k,l}(s) = \sum_{n=1}^{\infty} f_k(n, l) n^{-s} = \zeta_l(s)^k. \quad (2.5)$$

By uniqueness of the coefficients of the Dirichlet series, equation (2.5) can serve as a definition of $f_k(n, l)$ (see [5], for example).

In some circumstances it might be useful to use the hyperbola method (cf. [17, Theorem I.3.1]) for concrete calculation of $F_k(x, l)$. For $uv = x$, $l \geq 2$ and $0 \leq j \leq k$, we use $f_k(n, l) = f_{k-j}(n, l) * f_j(n, l)$ and (2.4) to get

$$F_k(x, l) = \sum_{i=1}^u F_{k-j}(x/i, l) f_j(i, l) + \sum_{i=1}^v F_j(x/i, l) f_{k-j}(i, l) - F_j(u, l) F_{k-j}(v, l).$$

This allows, for example, an efficient calculation of $F_{2k}(n, l)$ if $F_k(i, l)$ for $i = 1, \dots, n$ is already known:

$$F_{2k}(n, l) = 2 \sum_{i=1}^{\lfloor \sqrt[2k]{n} \rfloor} F_k(n/i, l) f_k(i, l) - F_k(\lfloor \sqrt[2k]{n} \rfloor, l)^2.$$

Another useful special case is the relation $F_2(n, l) = 2 \sum_{i=1}^{\lfloor \sqrt{n} \rfloor} \lfloor n/i \rfloor - \lfloor \sqrt{n} \rfloor^2 + (l-1)^2$.

The following theorem covers the recursive structure of the functions $f_k(n, l)$ and $F_k(x, l)$.

Theorem 2.1. *For $x, n \geq 1$ and $k, l \geq 1$ we have*

$$f_k(n, l) = \sum_{\substack{i=0 \\ l^i | n}}^k \binom{k}{i} f_{k-i} \left(\frac{n}{l^i}, l+1 \right) \quad (2.6)$$

$$F_k(x, l) = \sum_{i=0}^k \binom{k}{i} F_{k-i} \left(\frac{x}{l^i}, l+1 \right). \quad (2.7)$$

Further, for x, n, l as above and $k \geq 2$ we have

$$f_k(n, l) = \sum_{m=l}^{\lfloor \sqrt[2k]{n} \rfloor} \sum_{\substack{i=1 \\ m^i | n}}^k \binom{k}{i} f_{k-i} \left(\frac{n}{m^i}, m+1 \right) \quad (2.8)$$

$$F_k(x, l) = \sum_{m=l}^{\lfloor \sqrt[2k]{x} \rfloor} \sum_{i=1}^k \binom{k}{i} F_{k-i} \left(\frac{x}{m^i}, m+1 \right). \quad (2.9)$$

Proof. We first give a combinatoric proof of (2.6) and (2.7). The basic idea is the separation of factors equal to l . For fixed n, k, l and $0 \leq i \leq k$, we denote by $f_{k,i}(n, l)$ the number of factorizations of n , where all k factors are greater or equal to l and exactly i factors are equal to l . If l^i divides n , we have

$$f_{k,i}(n, l) = \binom{k}{i} f_{k-i} \left(\frac{n}{l^i}, l+1 \right),$$

because every factorization counted by $f_{k,i}(n, l)$ can be split into i factors equal to l and $k - i$ factors greater or equal to $l + 1$.

A similar argument gives

$$F_{k,i}(x, l) = \binom{k}{i} F_{k-i} \left(\frac{x}{l^i}, l + 1 \right)$$

for $0 \leq i \leq k$, where $F_{k,i}(x, l) := \sum_{n \leq x} f_{k,i}(n, l)$ counts all factorizations of integers less or equal to x , with k factors, where i factors are equal to l and $k - i$ are greater or equal to $l + 1$. Finally we get (2.6) and (2.7) from $f_k(n, l) = \sum_{i=0}^k f_{k,i}(n, l)$ and $F_k(x, l) = \sum_{i=0}^k F_{k,i}(x, l)$.

We proceed to show (2.9), by subsequent elimination of the first term of the right hand side of (2.7). More precisely, we separate the first term in the sum of (2.7) and apply (2.7) again (with $l + 1$ as second argument of $F_k(\cdot)$) to this term to get

$$\begin{aligned} F_k(x, l) &= F_k(x, l + 1) + \sum_{i=1}^k \binom{k}{i} F_{k-i} \left(\frac{x}{l^i}, l + 1 \right) \\ &= \sum_{i=0}^k \binom{k}{i} F_{k-i} \left(\frac{x}{(l+1)^i}, l + 2 \right) + \sum_{i=1}^k \binom{k}{i} F_{k-i} \left(\frac{x}{l^i}, l + 1 \right) \\ &= F_k(x, l + 2) + \sum_{m=l}^{l+1} \sum_{i=1}^k \binom{k}{i} F_{k-i} \left(\frac{x}{m^i}, m + 1 \right). \end{aligned}$$

Repeating the above operation j -times yields

$$F_k(x, l) = F_k(x, l + j + 1) + \sum_{m=l}^{l+j} \sum_{i=1}^k \binom{k}{i} F_{k-i} \left(\frac{x}{m^i}, m + 1 \right).$$

Setting $j = \lfloor \sqrt[k]{n} \rfloor - l$ and using (2.1) we get (2.9).

An analogous argument yields (2.8). This completes the proof. \square

For practical purposes, the performance of the recursions of theorem 2.1 is in most parameter constellations much better than the performance of the recursions (2.3) and (2.4). However, for large values of n, x the recursions tend to be numerically unstable.

The case $l = 1$ of theorem 2.1 connects $F_k(x)$ and $D_k(x)$, respectively $f_k(n)$ and $d_k(n)$.

Corollary 2.1. *For $x, n \geq 1$ and $k \geq 0$ we have*

$$D_k(x) = \sum_{i=0}^{\lfloor \log_2 x \rfloor} \binom{k}{i} F_i(x) \quad (2.10)$$

$$F_k(x) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} D_i(x) \quad (2.11)$$

$$d_k(n) = \sum_{i=0}^{\lfloor \log_2 n \rfloor} \binom{k}{i} f_i(n) \quad (2.12)$$

$$f_k(n) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} d_i(n). \quad (2.13)$$

Proof. The relations (2.10) and (2.12) follow directly from (2.7) and (2.6) with $l = 1$ and the boundary conditions (2.1).

By the definition of the binomial transform, we can say that for fixed $x \geq 1$ (resp. $n \geq 1$), $D_k(x)$ (resp. $f_k(n)$) is the binomial transform (with the respect to k) of $F_k(x)$ (resp. $f_k(n)$). Therefore the relations (2.11) and (2.13) can be deduced from the inversion of the binomial transform in general. \square

Remark 1: The relationship between $f_k(n)$ and $d_k(n)$ covered by corollary 2.1 seems to be well known, for example equation (2.13) is mentioned in [4, Chapter 17.2]. Equation (2.12) appears in a footnote of [16].

Remark 2: In this paper, we restrict ourselves to the case of factorizations where all integers greater or equal to a given l are allowed, since we are mainly interested in the case $l = 1$ and $l = 2$. The above formulas in theorem 2.1 and corollary 2.1 could be generalized to the case of factorizations consisting of arbitrary subsets of the positive integers (with at least two elements), as treated in [6] or [18]. The main idea in the proof of theorem 2.1 is to separate the smallest factor in the factorizations, which is also possible in the general (constrained) case. Similar results as in corollary 2.1 hold whenever 1 is the (smallest) element of the set of admissible factors.

Another remarkable relation between $f_k(n)$ and $d_k(n)$ is treated in the next corollary.

Corollary 2.2. *For $n \geq 1$ and $|u| > 1$ we have*

$$\sum_{k=0}^{\infty} u^{-k} d_k(n) = \frac{u}{u-1} \sum_{k=0}^{\lfloor \log_2 n \rfloor} (u-1)^{-k} f_k(n) \quad (2.14)$$

Proof. Recall that for given $k \geq 1$ the generating function of the binomial coefficients is given by

$$\sum_{i=k}^{\infty} \binom{i}{k} y^i = \frac{y^k}{(1-y)^{k+1}}, \quad (2.15)$$

with absolute convergence for $|y| < 1$.

For $n \geq 1$, $|y| < 1$ and large N , we have by (2.12) and lemma 3.2 (with $r = 0$) below

$$\begin{aligned} \sum_{k=0}^N y^k d_k(n) &= \sum_{k=0}^N y^k \sum_{i=0}^k \binom{k}{i} f_i(n) \\ &= \sum_{k=0}^N f_k(n) \sum_{i=k}^N \binom{i}{k} y^i. \end{aligned} \quad (2.16)$$

Using (2.15), by absolute convergence we can let $N \rightarrow \infty$ in (2.16) to get

$$\sum_{k=0}^{\infty} y^k d_k(n) = \sum_{k=0}^{\infty} \frac{y^k}{(1-y)^{k+1}} f_k(n)$$

Finally, we set $u := \frac{1}{y}$ and the claim follows by factoring out $\frac{1}{1-y} = \frac{u}{u-1}$ and taking into account $\frac{y}{1-y} = (u-1)^{-1}$. \square

Note that in (2.14) $d_k(n)$ and $f_k(n)$ can be replaced by $D_k(x)$ and $F_k(x)$, for $x \geq 1$, by the definition of $F_k(x, l)$ as the cumulated sum over $f_k(n, l)$, $n \leq x$.

Special cases of (2.14) include the equation $2f(n) = \sum_{k=0}^{\infty} 2^{-k} d_k(n)$ for $u = 2$. This formula was proved by Sen in [15] for the special case of square free n and then later used by Sklar in [16] to derive an asymptotic for $f(n)$ in this case.

3 Factorizations with a large number of factors

Throughout this section we use the notation $t = t(x, l) = \lfloor \log_l x \rfloor$ for given x and l . In this section (2.7) will be applied to show that $F_{t-j}(x, l)$ is a polynomial in t ; we give explicit formulas for the degree τ and the coefficients of the polynomial.

We begin by preparing two lemmas. The first lemma exploits the fact that $F_k(n, l)$ vanishes for large k and gives an explicit expression for the number of summands in (2.7).

Lemma 3.1. *For $x \geq 1$, $l \geq 1$, $k \geq 1$ we have*

$$F_k(x, l) = \sum_{i=0}^{\tau(x, k, l)} \binom{k}{i} F_i\left(\frac{x}{l^{k-i}}, l+1\right), \quad \text{with} \quad (3.1)$$

$$\tau(x, k, l) = \min\left(k, \left\lceil \frac{\log x - k \log l}{\log(l+1) - \log l} \right\rceil\right). \quad (3.2)$$

Proof. First note that in (3.1) we have reversed the order of summation in comparison to (2.7) and used the fact that $\binom{k}{i} = \binom{k}{k-i}$. From (2.1), the term $F_i(\frac{x}{l^{k-i}}, l+1)$ vanishes if either

$$l+1 > \sqrt[i]{n/l^{k-i}} \quad \text{or} \quad i > \log_{l+1}(n/l^{k-i}).$$

After some algebra, this leads in both cases to

$$i > \frac{\log n - k \log(l+1)}{\log l - \log(l+1)}.$$

This completes the proof. \square

The next lemma was already used in the proof of corollary 2.2.

Lemma 3.2. *For real ν_i , $\xi_{i,j}$ and γ_j , we have for $k \geq 1$ and $0 \leq r \leq k$*

$$\sum_{i=r}^k \nu_i \sum_{j=0}^{i-r} \xi_{i,j} \gamma_j = \sum_{j=0}^{k-r} \gamma_j \sum_{i=j+r}^k \nu_i \xi_{i,j}.$$

Proof. We write out the left hand side of the equation and rearrange terms to get

$$\begin{aligned} \sum_{i=r}^k \nu_i \sum_{j=0}^{i-r} \xi_{i,j} \gamma_j &= \nu_r (\xi_{r,0} \gamma_0) + \nu_{r+1} (\xi_{r+1,0} \gamma_0 + \xi_{r+1,1} \gamma_1) + \cdots + \nu_k (\xi_{k,0} \gamma_0 + \cdots + \xi_{k,k-r} \gamma_{k-r}) \\ &= \gamma_0 (\nu_r \xi_{r,0} + \cdots + \nu_k \xi_{k,0}) + \gamma_1 (\nu_{r+1} \xi_{r+1,1} + \cdots + \nu_k \xi_{k,1}) + \cdots + \gamma_{k-r} (\nu_k \xi_{k,k-r}) \end{aligned}$$

and the claim follows. \square

The next theorem is a straightforward implication of (2.7) and the fact that for positive integers $n \geq k$, we have $\binom{n}{k} = \sum_{i=0}^k n^i \left[\begin{smallmatrix} k \\ i \end{smallmatrix} \right] \frac{1}{k!}$, where $\left[\begin{smallmatrix} k \\ i \end{smallmatrix} \right]$ denotes the Stirling numbers of the first kind.

Theorem 3.1. *For $x \geq 1$, $l \geq 1$, $t = \lfloor \log_l x \rfloor$ and $j \leq t - 1$, we have $F_{t-j}(x, l) = P_\tau(t - j)$, where P_τ is a polynomial of degree τ in $t - j$ given by*

$$\begin{aligned} P_\tau(t - j) &= \sum_{m=0}^{\tau} v_m (t - j)^m, \quad \text{with} \\ \tau(x, j, l) &= \min \left(t - j, \left\lceil \frac{\{\log_l x\} + j \log l}{\log(l + 1) - \log l} \right\rceil \right) \\ v_m &= \sum_{i=m}^{\tau} \kappa_i \left[\begin{smallmatrix} i \\ m \end{smallmatrix} \right] \frac{1}{i!} \\ \kappa_i &= F_i(x_i, l + 1) \\ x_i &= l^{\{\log_l x\}} l^{i+j}, \quad i = 0, \dots, \tau. \end{aligned}$$

Proof. It follows from (3.1) that

$$\begin{aligned} F_{t-j}(x, l) &= \sum_{i=0}^{\tau(x,j,l)} \binom{t-j}{i} F_i \left(\frac{x}{l^{t-j-i}}, l + 1 \right), \quad \text{with} \\ \tau(x, j, l) &= \min \left(t - j, \left\lceil \frac{\log x - (t - j) \log l}{\log(l + 1) - \log l} \right\rceil \right). \end{aligned}$$

The expression for $\tau(x, j, l)$ in the theorem follows from

$$\log x - (t - j) \log l = \log x - (\lfloor \log_l x \rfloor - j) \log l = \{\log_l x\} + j \log l.$$

The first argument of F_i in the sum becomes

$$\frac{x}{l^{t-j-i}} = \frac{x}{l^{\lfloor \log_l x \rfloor - j - i}} = l^{\log_l x} l^{-\lfloor \log_l x \rfloor} l^{i+j} = l^{\{\log_l x\}} l^{i+j} = x_i,$$

so that we get $F_{t-j}(x, l) = \sum_{i=0}^{\tau(x,j,l)} \binom{t-j}{i} \kappa_i$, with $\tau(x, j, l)$ and $\kappa_i = F_i(x_i, l+1)$ as required. Next, we use the fact that the binomial coefficients in this expression can be written as a polynomial

$$\binom{t-j}{i} = \sum_{m=0}^i (t-j)^m \begin{bmatrix} i \\ m \end{bmatrix} \frac{1}{i!},$$

so that we get

$$\begin{aligned} F_{t-j}(x, l) &= \sum_{i=0}^{\tau} \kappa_i \sum_{m=0}^i (t-j)^m \begin{bmatrix} i \\ m \end{bmatrix} \frac{1}{i!} \\ &= \sum_{m=0}^{\tau} (t-j)^m \sum_{i=m}^{\tau} \kappa_i \begin{bmatrix} i \\ m \end{bmatrix} \frac{1}{i!}, \end{aligned}$$

where we have used lemma 3.2 (with $r = 0$) in the last equation. This completes the proof. \square

The calculation of the coefficients v_m of the polynomial in theorem 3.1 requires the calculation of Stirling numbers of the first kind and of values of $F_k(n, m)$ for parameters $m \geq l$, which is, in principle, possible via theorem 2.1. An easier method to derive the coefficients is given in the next corollary.

Corollary 3.1. *For given x, l and j as in theorem 3.1, we set $\lambda_i := F_i(x_i, l)$, with x_i ($i = 0, \dots, \tau$) and τ defined as in theorem 3.1. Let λ be the vector of the λ_i 's and B be the matrix of $b_{i,m} = (i+j)^m$ for $i, m = 0, \dots, \tau$. Then*

$$F_{t-j}(x, l) = \sum_{m=0}^{\tau} w_m t^m, \tag{3.3}$$

where the coefficients w_m are the elements of the vector $w = \lambda B^{-1}$ and $t = \lfloor \log_l x \rfloor$.

Proof. For given j and l , we know from theorem 3.1 that $F_{\lfloor \log_l y \rfloor - j}(y, l) = \sum_{m=0}^{\tau} w_m \lfloor \log_l y \rfloor^m$ for some coefficients w_m for all $y \geq 1$, where the w_m depend only on the value of $\{\log_l y\}$. Therefore, for a given x , we can choose $x_i = l^{\{\log_l x\} + i + j}$, for $i = 0, \dots, \tau$, with

$$F_i(x_i, l) = \sum_{m=0}^{\tau} w_m (i+j)^m$$

since $\lfloor \log_l x_i \rfloor = i + j$. Defining the vectors λ, w and the matrix B as in the corollary, the above equation reads $\lambda = wB$. Since B is invertible, we finally get $w = \lambda B^{-1}$.

This completes the proof. \square

Example 1: We calculate $F_{329}(10^{100})$ based on the above formulas. We have $k = 329$, $t = \lfloor \log_2 10^{100} \rfloor = 332$ and therefore $j = 3$. Lemma 3.1 gives $\tau = 5$.

Next we calculate, according to theorem 3.1, $\kappa_i = F_i(x_i, 3) = (1, 16, 36, 32, 15, 1)$, with $[x_i] = (9, 18, 36, 73, 146, 292)$ for $i = 0, \dots, 5$. With these values, we proceed to calculate the coefficients v_i of the polynomial in k .

With corollary 3.1, we calculate $\lambda_i = F_i(x_i) = (1, 17, 69, 189, 424, 837)$ for x_i as above and proceed to calculate the coefficients w_i of the polynomial in t .

Finally we get the following two polynomials

$$\begin{aligned} F_{329}(10^{100}) &= \frac{1}{120}k^5 + \frac{13}{24}k^4 + \frac{15}{8}k^3 + \frac{203}{24}k^2 + \frac{307}{60}k + 1 \\ &= \frac{1}{120}t^5 + \frac{5}{12}t^4 - \frac{31}{8}t^3 + \frac{223}{12}t^2 - \frac{252}{15}t + 53 \\ &= 38,535,596,289. \end{aligned}$$

Example 2: By calculating the polynomials at $n = 2^m$ and $n = 2^{m+1} - 1$ for $m = 0, 1, \dots$ and $l = 2$ with corollary 3.1, we can get explicit lower and upper bounds for $F_{t-j}(n)$, using the monotonicity of $F_k(\cdot)$:

$$\begin{aligned} 1 &\leq F_{t-0}(n) \leq t + 1 \\ 2t - 1 &\leq F_{t-1}(n) \leq \frac{1}{6}t^3 + \frac{3}{2}t^2 - \frac{2}{3}t \\ \frac{1}{6}t^3 + \frac{3}{2}t^2 - \frac{14}{3}t + 3 &\leq F_{t-2}(n) \leq \frac{1}{120}t^5 + \frac{1}{24}t^4 + \frac{49}{24}t^3 - \frac{253}{24}t^2 + \frac{449}{20}t - 19. \end{aligned}$$

4 An average order of $f_k(n)$

An average order of $f_k(n)$ is given by Hwang in [5, Corollary 3] as

$$F_k(x) = x \frac{(\log x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{k^2}{\log x}\right) \right), \quad (4.1)$$

with $1 \leq k = o((\log x)^{2/3})$. Lau [12, Theorem 2] was able to improve the error to $O(x^{1-\alpha_k}(\log x)^{k-1})$, with $\alpha_k = \epsilon k^{-2/3}$, for some $\epsilon > 0$ and $1 \leq k \leq ((\log x)^{3/5})$, but in his formula the main term is only specified up to some unknown constants. Note that in both approaches the parameter k is allowed to grow with x . We treat the easier case of finite values for k here.

Our approach to determine an average order of $f_k(n)$ for fixed k relies on the fact that for fixed n , $f_k(n)$ is the (inverse) binomial transform of $d_k(n)$, see (2.11) and (2.13). For the average order of $d_k(n)$, the following theorem is known, see [7, Chapter 13].

We use the notation $t = \log x$ for the rest of this section.

Theorem 4.1. *For $k \geq 1$, $\epsilon > 0$ and $\alpha_k = \frac{k-1}{k}$, there exist $a_{k,j}$, $j = 0, \dots, k-1$ with*

$$D_k(x) = xP_k^D(\log x) + \Delta_k^D(x), \text{ where} \quad (4.2)$$

$$P_k^D(t) = \sum_{j=0}^{k-1} a_{k,j} t^j \quad (4.3)$$

$$\Delta_k^D(x) = O(x^{\alpha_k + \epsilon}). \quad (4.4)$$

Note that for $k = 1$ we have $D_1(x) = [x]$ by (2.2) and therefore $\Delta_1^D(x) \leq 1$, with $a_{1,0} = 1$.

Explicit formulas for the coefficients $a_{k,j}$ ($k \geq 2$) of the main term as functions of the Stieltjes constants are given in [13]. The leading terms are given by $a_{k,k-1} = \frac{1}{(k-1)!}$. The estimation of the error term is known as the (Dirichlet) divisor problem. The currently best known values for the exponents α_k are given in [7]. It is conjectured that $\alpha_k = \frac{k-1}{2k}$ holds.

With this preparation, we are able to prove the following theorem for the average order of $f_k(n)$.

Theorem 4.2. *For $k \geq 1$ and $\epsilon > 0$ we have $F_k(x) = xP_k^F(\log x) + \Delta_k^F(x)$, where*

$$P_k^F(t) = \sum_{j=0}^{k-1} b_{k,j} t^j \quad (4.5)$$

$$b_{k,j} = \sum_{i=j+1}^k (-1)^{k-i} \binom{k}{i} a_{i,j} \quad (4.6)$$

$$\Delta_k^F(x) = O(x^{\beta_k + \epsilon}) \quad (4.7)$$

$$\beta_k = \max_{1 \leq j \leq k} \alpha_j. \quad (4.8)$$

Proof. First note that for $k = 1$ by (2.2) we have $F_1(x) = (\lfloor x \rfloor - 1)^+$ and the claim follows. Let $\epsilon > 0$ and $k \geq 2$ be given. From (2.11) and theorem 4.1 we get

$$F_k(x) = (-1)^k + \sum_{i=1}^k (-1)^{k-i} \binom{k}{i} (xP_i^D(\log x) + \Delta_i^D(x))$$

and therefore $F_k(x) = xP_k^F(\log x) + \Delta_k^F(x)$ with

$$P_k^F(t) = \sum_{i=1}^k (-1)^{k-i} \binom{k}{i} \sum_{j=0}^{i-1} a_{i,j} t^j$$

$$|\Delta_k^F(x)| \leq \left| \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} C_{\epsilon,i}^{(D)} x^{\alpha_i + \epsilon} \right|$$

for constants $C_{\epsilon,i}^{(D)} > 0$ and x large enough (the term $(-1)^k$ is asymptotically negligible). Applying lemma 3.2 (with $r = 1$) to the first equation gives formula (4.6) for the coefficients of the P_k^F -polynomial.

Defining β_k as in (4.8) and $C_{\epsilon,k}^{(F)} := \sum_{i=0}^k \binom{k}{i} C_{\epsilon,i}^{(D)}$, we get $|\Delta_k^F(x)| \leq C_{\epsilon,k}^{(F)} x^{\beta_k + \epsilon}$, which proves (4.7). \square

Note that the coefficients of the leading term in the P_k^F -polynomial are given by $b_{k,k-1} = \frac{1}{(k-1)!}$ and therefore the leading term coincides with the main term in (4.1).

For $k = 2$ and $x \leq 2 \cdot 10^7$, we found that $|\Delta_2^F(x)| < 356.1$, where the maximum value was reached at $x_{max} = 19,740,240$ with $F_2(x_{max}) = 334,648,770$.

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